Problem 1

Let \( f(x) \) be given by:
\[
\begin{align*}
    f(x) = \begin{cases} 
    2x - 4\tan(x) & \text{if } x \leq 0 \\
    x^a + x^2 & \text{if } 0 < x < 1 \\
    9x - 7 & \text{if } x \geq 1
    \end{cases}
\end{align*}
\]
where \( a \) is a real number.

(a) [5 points] Is the function continuous at \( x = 0 \)? Why or why not?

\[
\begin{align*}
    \alpha \to x = 0, \quad f(x) &= 2 \cdot 0 + 4 \cdot \tan(0) = 0 \\
    \lim_{x \to 0^+} x^a + x^2 &= 0 \cdot a + 0^2 = 0 \quad \text{(if } a > 0), \text{ so} \\
    f(x) \text{ is continuous if } a > 0.
\end{align*}
\]

When \( a < 0 \),
\[
\lim_{x \to 0^+} x^a + x^2 = \lim_{x \to 0^+} \frac{1}{x^{a-1}} + x^2
\]
is undefined.

Therefore, \( f(x) \) is not continuous when \( a < 0 \).

(b) [5 points] For what values of \( a \) (if one exists) is \( f(x) \) differentiable at \( x = 1 \)? (Hint: power rule) For each of those values of \( a \), find \( f'(1) \).

At \( x = 1 \),
\[
\begin{align*}
    x^a + x^2 &= 1 + 1^2 = 1 + 1 = 2 \\
    a \cdot 1 - 7 &= a \cdot 1 - 7 = 2
\end{align*}
\]
So \( f(x) \) is continuous.

\[
\begin{align*}
    \lim_{x \to 1^-} f'(x) &= \lim_{x \to 1^-} (x^a + x^2)' = \lim_{x \to 1^-} a \cdot x^{a-1} + 2x = a \cdot 1^{a-1} + 2 \cdot 1 = 2 + a \\
    \lim_{x \to 1^+} f'(x) &= \lim_{x \to 1^+} (9x - 7)' = 9 - 7 = 2
\end{align*}
\]

\[
2 + a = 2 \implies a = 0 \quad \text{and} \quad f'(1) = 2
\]
Problem 2

(a) [5 points] Solve the following trigonometry equation for $0 \leq \theta \leq 2\pi$:

$$2 \cos^2 \theta - \cos \theta = 1$$

Let $x = \cos \theta$, then $2x^2 - x = 1$

$$\Rightarrow 2x^2 - x - 1 = 0 \Rightarrow (2x + 1)(x - 1) = 0$$

$$\Rightarrow x = -\frac{1}{2} \quad \text{or} \quad x = 1$$

when $\cos \theta = -\frac{1}{2}$, $\theta = \frac{2\pi}{3}$, or $\frac{4\pi}{3}$

when $\cos \theta = 1$, $\theta = 0$, or $2\pi$

(b) [5 points] Use the limit definition of derivative to compute the derivative of $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} 2x + h = 2x$$
Problem 3

Find the derivative $\frac{dy}{dx}$ for each of the following equation. (a) [5 points] $y = \frac{1}{x^2+1}$

$$y' = \left( \left( x^2 + 1 \right)^{-1} \right)' = (-1) \cdot (x^2 + 1) \cdot 2x$$

$$= -2x(x^2 + 1)$$

(b) [5 points] $y = \sin(2x^2)$

$$y' = \cos(2x^2) \cdot 4x$$

$$= 4x \cos(2x^2)$$
(c) [5 points] \[ x^2 + y^2 = 4 \]

\[ \frac{d}{dx} \left[ x^2 + y^2 \right] = \frac{d}{dx} [4] \]

\[ \Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0 \]

\[ \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \]

(d) [5 points]

\[ \frac{xy - y^2}{y - x} = 4 \]

\[ \Rightarrow \frac{y(x - y)}{y - x} = 4 \]

\[ \Rightarrow y = 4 \quad (x \neq y) \]

\[ \Rightarrow \frac{dy}{dx} = 0 \]
Problem 4
(a) [6 points] Find the differential $dy$ for the following function.

\[ y = -(x - 1)^3(x - 5) \]

\[ dy = \left[ -3(x - 1)^2(x - 5) \right] \cdot dx \]

\[ = \left( -2(x - 1)^2 \cdot (3x - 15 + x - 1) \right) \cdot dx \]

\[ = -2(x - 1)^2 \left( 4x - 16 \right) \cdot dx \]

\[ = -4\left( x - 1 \right)^2 \left( y - 4 \right) \cdot dx \]

(b) [6 points] Find the vertical and horizontal asymptotes of

\[ f = \frac{x^2 - 2x - 24}{x^2 + x - 12} = \frac{(x+4)(x-6)}{(x-3)(x+4)} \]

Vertical asymptotes:

\[ \lim_{x \to 3^-} \frac{x-6}{x-3} = \frac{3-6}{0^-} = -\infty \]

\[ \lim_{x \to 3^+} \frac{x-6}{x-3} = \frac{3-6}{0^+} = +\infty \]

Horizontal asymptotes:

\[ \lim_{x \to \infty} \frac{x-6}{x-3} = \lim_{x \to \infty} \frac{x-6}{x-3} = \frac{1-\frac{6}{x}}{1-\frac{3}{x}} = 1 \]

(c) [6 points] Find the second derivative of $y = x^2 \cos(3x)$

\[ y' = 2x \cdot \cos(3x) + x^2 \cdot \left( -\sin(3x) \cdot 3 \right) \]

\[ = 2x \cdot \cos(3x) - 3x^2 \cdot \sin(3x) \]

\[ y'' = 2 \cdot \cos(3x) + 2x \cdot \left( -\sin(3x) \cdot 3 \right) - 6x \cdot \sin(3x) \]

\[ = -12x \cdot \sin(3x) + 2 \cos(3x) - 9x^2 \cdot \cos(3x) \]
Problem 5
(a) [6 points] The surface area of a cube is increasing at a rate of 4 in$^2$/sec. How fast is the volume changing when the length of each edge is 2 inches long?

(b) [6 points] Find the equation for the line that pass through the point (0, 0) and is parallel to the tangent of $f(x) = x^2$ at $x = 1/2$. 
Problem 6

[10 points]
The profit derived from selling \( x \) units of a particular product is modeled by the formula

\[ P = 16x^2 + 30x - 170. \]

(a) [3 points] Find the formula for the differential \( dP \).

\[ dP = \left( 32x + 30 \right) \, dx \]

(b) [3 points] What is the actual gain in profit obtained by increasing the sales from 20 to 21 units?

\[ \Delta P = \left( 16 \left( 21 \right)^2 + 30 \cdot 21 - 170 \right) \]
\[ - \left( 16 \cdot (20)^2 + 30 \cdot 20 - 170 \right) \]
\[ = 16 \left( 21^2 - 20^2 \right) + 30 \cdot (21 - 20) \]
\[ = 16 \cdot (21 + 20) \cdot (21 - 20) + 30 \]
\[ = 16 \times 41 + 30 = 656 + 30 = 686 \]

(c) [4 points] Use the differential \( dP \) to approximate the change in profit when the sales increase from 20 to 21 units.

\[ dP = \left( 32 \cdot 20 + 30 \right) \cdot 1 \]
\[ = 640 + 30 = 670 \]
Problem 7 (Multiple-Choices/True-and-False Questions) [20 points]

Please circle one and only one answer for problems 1-5.

(1) \[ \lim_{x \to \pi^-} \frac{\cos x}{x} = \]
(A) -\(\pi\)  (B) 1/\(\pi\)  (C) \(\infty\)  (D) Do not exist  (E) None of the above

(2) The domain of \( f(x) = 1/\sqrt{1-x^2} \) is
(A) all real numbers  (B) all real numbers except for \( x = 1 \) or \( x = -1 \)
(C) \(-1 \leq x \leq 1\)  (D) \(-1 < x < 1\)

(3) If \( y = \sin^2(4x) \) then \( y'(\frac{\pi}{12}) = \)
(A) 8  (B) 0  (C) 4\(\sqrt{3}\)  (D) 2\(\sqrt{2}\)  (E) None of the above

(4) \[ \lim_{x \to 3} \frac{x^2 - 9}{x + 3} = \]
(A) \(\infty\)  (B) 2  (C) \(-\infty\)  (D) -2  (E) None of the above

(5) The curve \( y = x^2 + 10x \) has a horizontal tangent line when \( x \) is
(A) 5  (B) -5  (C) 10  (D) -10  (E) None of the above

Please read carefully each statement in Problems 6-10, and determine whether each is True or False.

(6) The product rule is \( (f \cdot g)' = f' \cdot g + f \cdot g' \).
☐ True  ☐ False

(7) If \( f(x) = x^2 + 1 \) and \( g(x) = \sqrt{x} \). Then \( f(g(x)) = x + 1 \).
☐ True  ☐ False

(8) If \( (c, f(c)) \) is a point of inflection of the graph of \( f \), then the derivative of \( f \) must be zero.
☐ True  ☐ False

(9) \( y = x^3 \) has an inflection point at \( (0, 0) \).
☐ True  ☐ False

(10) The second derivative of \( f(x) = \cos(x) \) is \( f''(x) = \tan(x) \).
☐ True  ☐ False